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Quantum Statistics of Ideal Gases in Two Dimensions

ROBERT M. MAY
The Daily Telegraph Theoretical Department, School of Physics, University of Sydney, Sydney, Australia
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It is shown that in two dimensions the specific heat $C_V(T,N)$ for an ideal gas of Fermi particles is identical with that for an ideal Bose gas for all T and N. This is true despite the great difference in the distribution functions of the two systems at low temperatures. To shed further light on this identity, the quantum statistics of ideal gases are investigated treating the number of dimensions as a continuous variable: $n=2$ is seen to be a special case. In the extreme relativistic region, the analogous special case is $n=1$.

1. INTRODUCTION

THE thermodynamic properties of a system of noninteracting particles, obeying either Bose-Einstein or Fermi-Dirac statistics, are well known[1,2] in three dimensions. At high temperatures, both systems approach the classical limit, with a specific heat given by the equipartition theorem. At low temperatures, the Fermi particles crowd into the lowest energy configuration, subject to the Pauli principle: The result is that a block of the lowest energy states is filled uniformly, with one particle per state. There is no such restriction for the bosons, and at temperatures below a certain transition temperature a finite fraction of the total number of bosons condenses into the ground state: this behavior is reflected by a discontinuity in the slope of the specific heat.

In two dimensions it is well known that the Bose gas does *not* condense, and thus there is no discontinuity in its specific heat. In this paper we prove that in fact the specific heat $C_V(N,T)$ of an ideal gas of N bosons at temperature T is identical with that of an ideal gas of N fermions at temperature T, for all N and all T, in two dimensions. This is done in Sec. 2. The theorem is surprising at first sight, because at low temperatures the two distribution functions are very different. Although the bosons do not condense, they do crowd into the lowest few states, giving a sharply peaked distribution; the Fermi distribution is similar to the three-dimensional one with uniform occupation of all states up to the Fermi energy. (This is exemplified by comparing the behavior of a charged, two-dimensional Bose gas with that of a Fermi gas in a magnetic field at low temperatures: the two results are altogether different.[3])

The theorem serves as a striking example that the specific heat can be an unreliable guide to the distribution function of a system.

To shed further light on the above theorem, in Sec. 3 we consider the thermodynamic properties of ideal quantum gases, regarding the number of dimensions as a continuous variable n. If the usual high-temperature cluster expansion is made,[4] then for $n>2$ the results are similar to those for the three-dimensional case: $C_V(N,T)$ has the equipartition value, with first-order corrections which are positive for bosons and negative for fermions. The Fermi C_V decreases monotonically to zero at $T=0$; the Bose C_V must also reach zero at $T=0$, and this it does by means of a discontinuity in slope at a critical temperature, below which temperature the gas condenses. (For $n>4$ the Bose gas has a discontinuity in C_V.) $n=2$ is the unique case for which the first-order corrections to the equipartition value of C_V vanish. It is also the value of n for which the Bose gas just fails to condense. For $n<2$, the Bose C_V decreases monotonically to zero, while the Fermi C_V increases above the classical value but then decreases smoothly to zero at $T=0$.

Finally, in Sec. 4, we investigate the quantum gases in the extreme relativistic region, again regarding the number of dimensions as a continuous variable. It is shown that the behavior of the extreme relativistic gas in n dimensions has the character of the behavior of the nonrelativistic gas in $2n$ dimensions.

2. PROOF OF THEOREM ON $C_V(N,T)$

The grand canonical partition function, $\exp(-\alpha\Omega)$, for an ideal Bose gas (Ω_B) or an ideal Fermi gas (Ω_F) is given by

$$\Omega_{\substack{B\\F}}=\pm kT\sum_{\mathbf{k}}\ln[1\mp\exp(-\eta-\alpha E_{\mathbf{k}})]. \tag{1}$$

This formula is valid for an arbitrary number of dimensions: $\mathbf{k}$ labels the energy states; $\alpha\equiv 1/kT$; $\eta\equiv-\alpha\mu$, where μ is the chemical potential. All other thermodynamic functions may be derived from Ω: In particular, the number of bosons or fermions present is

$$N_{\substack{B\\F}}=\sum_{\mathbf{k}}\frac{1}{\exp(\eta+\alpha E_{\mathbf{k}})\mp 1}. \tag{2}$$

(For Bose statistics, negative occupation numbers are avoided by the demand $\eta>0$.) The relation (2) may be used to eliminate η in favour of N in (1).

For ideal particles in a cubical box with periodic

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[1] A. Einstein, Ber. Berlin Akad. 261 (1924).
[2] F. London, Phys. Rev. **54**, 947 (1938).
[3] R. M. May, Phys. Rev. **115**, 254 (1959).
[4] J. Mayer and M. Mayer, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1940).

boundary conditions, the energy values are

$$E_{\mathbf{k}}=\hbar^2k^2/2m;\quad \mathbf{k}=(2\pi/L)\boldsymbol{\nu}, \tag{3}$$

where $\boldsymbol{\nu}$ are the points in a unit cubical lattice. Provided $mkT/\hbar^2>L^{-2}$ we can replace sums by two-dimensional integrals in (1) and (2) to get

$$\Omega_{\substack{B\\F}}=\pm kT\frac{V}{\lambda^2}\int_0^\infty dx\,\ln[1\mp e^{-(\eta+x)}] \tag{4}$$

and

$$N_{\substack{B\\F}}=\frac{V}{\lambda^2}\int_0^\infty \frac{dx}{e^{\eta+x}\mp 1}; \tag{5}$$

that is,

$$N_{\substack{B\\F}}=\mp(V/\lambda^2)\ln(1\mp e^{-\eta}). \tag{6}$$

V is the two-dimensional "volume," and λ is the "thermal wavelength" defined by

$$\lambda^2=4\pi\alpha\hbar^2/2m. \tag{7}$$

Note that the integral in (5) for N_B has no upper bound (it diverges logarithmically as $\eta\rightarrow 0$), so that there is no condensation phenomenon for the two-dimensional Bose gas. However, we can see that at low temperatures the bosons will crowd into the low-lying states to give something like a momentum condensation (e.g., if charged, they give an "imperfect" Meissner effect which is practically indistinguishable from the perfect London one of the three-dimensional Bose gas). On the other hand, we see that the two-dimensional fermions at low temperatures behave just as three-dimensional fermions.

For convenience we define quantities $T_0(N)$ and $\tau(N,T)$:

$$N=V/\lambda_0^2;\quad \tau(N,T)=\exp(-T_0/T). \tag{8}$$

T_0 is a temperature which divides the classical regime ($T>T_0$ and $\tau\sim 1$) from the regime of quantum statistics ($T<T_0$ and $\tau\ll 1$). This statement is made clearer by rewriting the definition (8) as

$$\left(\frac{T}{T_0}\right)^{1/2}=\frac{\text{mean interparticle separation}}{\text{thermal wavelength}}. \tag{9}$$

The total energy of the system can be written from (4):

$$E_{\substack{B\\F}}=kT\frac{V}{\lambda^2}\int_0^\infty \frac{xdx}{e^{\eta+x}\mp 1}. \tag{10}$$

Expanding the denominator and integrating, and also using Eqs. (6) and (8) to eliminate η, we get

$$E_B(N,T)=Nk(T^2/T_0)F_+(1-\tau;2) \tag{11}$$

and

$$E_F(N,T)=Nk(T^2/T_0)F_-\left(\frac{1}{\tau}-1;2\right), \tag{12}$$

where the functions F_+ and F_- are defined in general by

$$F_\pm(z;m)=\sum_{i=1}^\infty(\pm 1)^{i+1}\frac{z^i}{i^m}. \tag{13}$$

We now prove a lemma, namely, the mathematical identity

$$F_-\left(\frac{1}{\tau}-1;2\right)=F_+(1-\tau;2)+\tfrac{1}{2}(\ln\tau)^2 \tag{14}$$

with $\tau<1$. To do this we note that F_+ and F_- can be written

$$F_\pm(z;2)=\mp\int_0^z\frac{\ln(1\mp t)}{t}dt. \tag{15}$$

Then, changing the variable of integration from t to $s=t/(1+t)$, we get

$$F_-\left(\frac{1}{\tau}-1;2\right)=-\int_0^{1-\tau}\frac{\ln(1-s)}{s(1-s)}ds \tag{16}$$

$$=-\int_0^{1-\tau}\frac{ds}{s}\ln(1-s)-\int_\tau^1\frac{dr}{r}\ln(r), \tag{17}$$

which establishes the identity (14).

Making use of (14) in (11) and (12), and remembering the definition of τ[cf. Eq. (8)], we can write

$$E_F(N,T)=E_B(N,T)+\tfrac{1}{2}NkT_0. \tag{18}$$

That is, in two dimensions, the Bose and Fermi distribution functions are just such that, for a given N and T, the total energies of the two systems differ by a quantity proportional to N and independent of T.

From the definition

$$C_V(N,T)=(dE/dT)_{N,V}, \tag{19}$$

it follows immediately that

$$[C_V(N,T)]_B=[C_V(N,T)]_F. \tag{20}$$

As a corollary it follows that a mixture of N_1 fermions and N_2 bosons has exactly the same specific heat curve as a system of N_1+N_2 fermions or N_1+N_2 bosons.

3. IDEAL GASES IN *n* DIMENSIONS

In order to gain further understanding of the results in Sec. 2, we now formulate the expression for the grand canonical potential Ω (and hence all other thermodynamic functions), regarding the number of dimensions in which the particles move as a continuous variable n.

To replace sums by integrals in Eq. (1) after the style of Sec. 2 [cf. Eq. (4)], we note that the corresponding density of states with wave numbers between k and $k+dk$ in n dimensions is

$$[V/(2\pi)^n]S(n)k^{n-1}dk, \tag{21}$$

where V is the n-dimensional "volume" and $S(n)$ is the surface area of the unit sphere in n dimensions

$$S(n)=2\pi^{n/2}/\Gamma(n/2). \tag{22}$$

Then, using the thermal wavelength defined by (7), we can write

$$\Omega_{\substack{B\\F}}=\pm kT\left(\frac{2m}{4\pi\alpha\hbar^2}\right)^{n/2}\frac{V}{\Gamma(n/2)}\times\int_0^\infty x^{(n/2)-1}dx\,\ln(1\mp e^{-\eta-x}) \tag{23}$$

$$=-kT\frac{V}{\lambda^n}F_\pm\left(e^{-\eta};\frac{n}{2}+1\right). \tag{24}$$

The F_+ and F_- functions are defined by (13). All other thermodynamic quantities can be derived from Ω; in particular, the number of particles is

$$N_{\substack{B\\F}}=\frac{V}{\lambda^n}\frac{1}{\Gamma(n/2)}\int_0^\infty\frac{x^{(n/2)-1}dx}{e^{\eta+x}\mp1} \tag{25}$$

$$=\frac{V}{\lambda^n}F_\pm\left(e^{-\eta};\frac{n}{2}\right). \tag{26}$$

Notice that for Fermi statistics, the integral for N_F can always be made arbitrarily large (provided that $n>0$) by making η large and negative. Thus, there is no upper bound to the expression for N_F, and hence no condensation phenomenon in ideal Fermi gases. On the other hand, for Bose statistics (remembering the restriction $\eta>0$) we see that the integral in (25) has an upper bound once $n>2$: for given N_B we can then define a critical temperature T_C,

$$N_B=(V/\lambda_C{}^n)\zeta(n/2);\quad (n>2). \tag{27}$$

[$\zeta(z)$ is the Riemann zeta function[5] of order z.] Above T_C all particles can be accommodated above the ground state; below T_C the replacement of sums by integrals in (1) is not valid, and the ground state contains a finite fraction of the total number of Bose particles:

$$N_0=N\{1-(T/T_C)^{n/2}\}. \tag{28}$$

Thus, we already see that $n=2$ is a special case in the continuum of dimensions for Bose statistics.

Next we take the usual high-temperature (classical) limit. A temperature T_0 is defined, in n dimensions, analogous to that defined by Eq. (8):

$$N=V/\lambda_0{}^n;\quad N\lambda^n/V=(T_0/T)^{n/2}. \tag{29}$$

Then as before T_0 divides the regime of classical behavior from the regime where quantum effects come into play. (Notice that for Bose statistics in $n>2$ dimensions, $T_0\sim T_C$.) For $T>T_0$ we can expand Ω and other thermodynamic quantities by use of the power series expansion (13). Also, we can use (26) to eliminate η in favor of N[i.e., in favor of $T_0(N)$] to get

$$\Omega_{\substack{B\\F}}=-NkT\left\{1\mp\frac{1}{2^{(n/2)+1}}\left(\frac{T_0}{T}\right)^{n/2}+\left(\frac{1}{2^n}-\frac{2}{3^{(n/2)+1}}\right)\left(\frac{T_0}{T}\right)^n+\cdots\right\}, \tag{30}$$

whence

$$[C_V]_{\substack{B\\F}}=\frac{n}{2}Nk\left\{1\pm\left(\frac{n}{2}-1\right)\frac{1}{2^{(n/2)+1}}\left(\frac{T_0}{T}\right)^{n/2}-(n-1)\left(\frac{1}{2^n}-\frac{2}{3^{(n/2)+1}}\right)\left(\frac{T_0}{T}\right)^n+\cdots\right\}. \tag{31}$$

The leading term in (31) is the classical equipartition value for C_V, with $\frac{1}{2}Nk$ for each degree of freedom. $n=2$ is obviously a special case in that it is the only value such that the first-order correction term vanishes. The second-order term is common to both statistics. For $n>2$ we have the situation familiar from three dimensions: The Fermi specific heat falls steadily to reach zero at $T=0$ [as it must: $C_V(T=0)=0$ is demanded by the third law of thermodynamics]. The Bose specific heat initially increases above the equipartition value—we can see that it increases down to T_C, then there is a discontinuity in the slope and C_V decreases to zero. For $n<2$, it is the Bose-gas specific heat which decreases monotonically to zero. The Fermi C_V must be an analytic function of T, and we can show that it rises above the classical value to attain a maximum value at $T\sim T_0$ and then falls smoothly to zero at $T=0$.

To conclude this section, we write down the expression for C_V which holds for all T except for the Bose gas below its critical temperature T_C [cf. (27)]:

$$\frac{[C_V]_{\substack{B\\F}}}{\frac{1}{2}nNk}=\left(\frac{n}{2}+1\right)\left[F_\pm\left(e^{-\eta};\frac{n}{2}+1\right)\Big/F_\pm\left(e^{-\eta};\frac{n}{2}\right)\right]-\left(\frac{n}{2}\right)\left[F_\pm\left(e^{-\eta};\frac{n}{2}\right)\Big/F_\pm\left(e^{-\eta};\frac{n}{2}-1\right)\right], \tag{32}$$

with η given as a function of N and T by Eq. (26). The exceptional case, the Bose gas for $T<T_C$, is easily seen to give

$$\frac{[C_V(T<T_C)]_B}{\frac{1}{2}nNk}=\left(\frac{n}{2}+1\right)\left[\zeta\left(\frac{n}{2}+1\right)\Big/\zeta\left(\frac{n}{2}\right)\right]\left(\frac{T}{T_C}\right)^{n/2}, \tag{33}$$

[5] See, for example, E. Whittaker and G. Watson, *Modern Analysis* (Cambridge University Press, London, 1946).

where $n>2$. It is clear that $[C_V]_F$ is always regular (for $n>0$) even though it is only monotonic for $n>2$. More interesting is the Bose gas for $n>2$: We can calculate the discontinuity in the specific heat at $T=T_C$ by taking the limits as T tends to T_C from above, $(C_V)_+$, and from below, $(C_V)_-$:

$$\frac{(C_V)_- - (C_V)_+}{\frac{1}{2}nNk}=\frac{n}{2}\left[\zeta\left(\frac{n}{2}\right)\Big/\zeta\left(\frac{n}{2}-1\right)\right]. \qquad (34)$$

Note that $\zeta(z)$ diverges for $z\leqslant 1$, so there is no discontinuity in C_V so long as

$$4\geqslant n(>2). \qquad (35)$$

For $n>4$ the Bose-Einstein condensation gives a thermodynamic transition of the first kind, with a finite discontinuity in C_V.

4. EXTREME RELATIVISTIC LIMIT

In the extreme relativistic limit, the energy levels for the ideal gas, in an n-dimensional cubic box with periodic boundary conditions, assume the form

$$E_{\mathbf{k}}=\hbar ck;\quad \mathbf{k}=(2\pi/L)\boldsymbol{\nu}, \qquad (36)$$

where $\boldsymbol{\nu}$ are the points in an n-dimensional unit cubic lattice. By this we mean that we are considering a problem in which particles with wave numbers such that $\hbar k<mc$ are to be regarded as being in the ground state. This is the case if the rest mass $m\to 0$, or in general if $kT\gg mc^2$. (In the earlier work in Secs. 2 and 3, the rest energy mc^2 was of course absorbed into the chemical potential μ.)

Putting (36) for $E_{\mathbf{k}}$ into Eq. (1), and replacing sums by n-dimensional integrals after the manner of Sec. 3, we get for the relativistic grand canonical potential

$$\Omega_{\substack{B\\F}}=\pm kT\frac{2}{(4\pi)^{n/2}}\frac{V}{\Gamma(n/2)}\times\int_0^\infty k^{n-1}dk\,\ln(1\mp e^{-\eta-\alpha\hbar ck}). \qquad (37)$$

That is to say,

$$\Omega_{\substack{B\\F}}=-kTA(n)(V/\lambda^{2n})F_{\pm}(e^{-\eta};n+1), \qquad (38)$$

where λ is the thermal wavelength as defined by (7), and $A(n)$ is defined as

$$A(n)=\frac{2\Gamma(n)}{\Gamma(n/2)}\left(\frac{\pi^{1/2}\hbar}{mc}\right)^n. \qquad (39)$$

Notice A is independent of N and T, involving only fundamental constants and n.

Equation (38) for Ω is just like Eq. (24) of Sec. 3, multiplied by a function of n [namely, $A(n)$] and with n in (24) replaced by $2n$. Since n is a constant for a given thermodynamic problem, *all* thermodynamic functions in the extreme relativistic limit in n dimensions will have the same form as the corresponding nonrelativistic functions in $(2n)$ dimensions. For example, we can write directly from Eq. (26) that

$$N_{\substack{B\\F}}=(V/\lambda^{2n})A(n)F_{\pm}(e^{-\eta};n) \qquad (40)$$

in the relativistic limit.

[The significance of the factor $A(n)$ is, of course, that the temperature which divides quantum from classical behavior is no longer T_0 but its relativistic analog, T_1 say, where

$$\frac{T}{T_1}\equiv\frac{\text{mean interparticle separation}}{\text{relativistic thermal wavelength}}. \qquad (41)$$

The relativistic thermal wavelength is given by the analog of Eq. (7): $\lambda_{\rm rel}\equiv c\hbar/kT$.]

With these remarks, together with the results of Secs. 2 and 3, we arrive directly at the following conclusions. The specific heat $C_V(N,T)$ of an extreme relativistic gas of N bosons at temperature T is identical with that of N fermions at temperature T, for all N and T, *in one dimension*. If the number of dimensions n per particle in the relativistic gas is regarded as a continuous variable, then $n=1$ is the analog of the nonrelativistic special case $n=2$. The relativistic Bose gas condenses for $n>1$, with a critical temperature T_C' given by

$$N_B=V\left(\frac{kT_{C'}}{2\pi^{1/2}\hbar c}\right)^n\frac{2\Gamma(n)}{\Gamma(n/2)}\zeta(n). \qquad (42)$$

Moreover, for $n>2$ this thermodynamic transition of the relativistic Bose gas is of the first kind, with a latent heat. (For $2\geqslant n>1$ it is of the second kind.) In particular, the discontinuity in three dimensions is given by putting $n=6$ in Eq. (34):

$$\frac{(C_V)_- - (C_V)_+}{3Nk}=3\frac{\zeta(3)}{\zeta(2)}=2.192 \qquad (43)$$

for the extreme relativistic Bose gas. (nNk is the relativistic equipartition specific heat in n dimensions.)

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